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The Hagia Sophia

A masterpiece of Byzantine architecture, the Hagia Sophia (in Greek, hagia = holy, sophia = wisdom) is one of the great buildings of the world. It was constructed in an incredibly short time between 532 and 537 during the reign of the Byzantine emperor Justinian. Its two architects were mathematicians and scientists, skilled in geometry and engineering. All of their talents were needed for the execution of the unprecedented design of this monumental church. The central part of the structure consists of four large semicircular arches arranged in a square and topped by a dome in the shape of a section of a hemisphere. Figure 3.1 depicts the essence of the design. The four curved triangular structures created by the circular base of the dome and the four arches are called

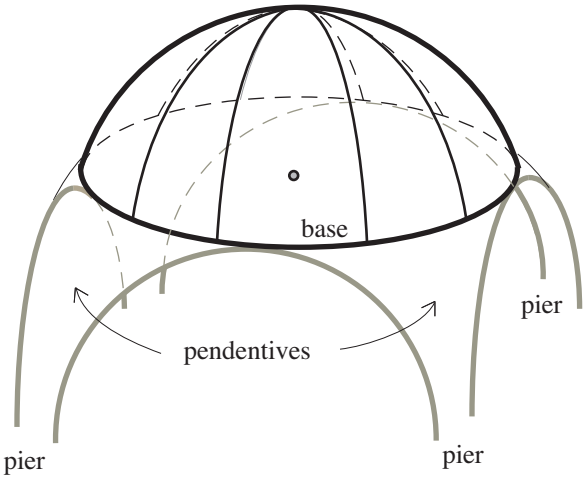

Figure 3.1. The dome of the Hagia Sophia in the abstract

pendentives. The massive supporting columns are known as *piers*. The dome of the Hagia Sophia is about 105 feet in diameter at its base and rises 180 feet above the floor at its highest point. One pair of the large arches under the dome open into half domes and in turn into recesses that together provide the church with a continuous clear space of 250 feet in length. The circular arcade of forty windows around the base of the dome gives the impression that the dome is floating above the soaring space that it creates. The interior surfaces were covered with marble, murals, and golden mosaics. The delicate use of color in the composition of the mosaic of Plate 6 tells us how sophisticated and splendid this art form would become. The interior with its domes, arches, and vaults had a celestial quality. The light that poured in from windows at many levels to touch the interior surfaces would have enriched their artistry and splendor. The elaborate religious ceremonies that this space hosted were officiated by the Greek Orthodox clergy in the sanctuary around the altar. The participation by the members of the imperial court near the entrance reflected the Byzantine duality of church and empire. The experience that any witness to these services would have had must have been spectacular. At around the year 1000, these witnesses included emissaries

Plate 6. Detail of the Deesis mosaic from the Hagia Sophia. (The word deesis comes from the Greek, meaning entreaty, as the icon is intended to ask for the intercession of Christ for humanity on judgement day). The mosaic is said to date from 1261. Because of the softness of the features of the face and its delicate hues it is regarded to be one of the finest mosaics of its time

sent by Prince Vladimir of Kiev to the centers of the four great religions of this region of the world: Islam, Judaism, and Latin and Byzantine Christianity. Their glowing description

> *We knew not whether we were in heaven or on earth. For on earth there is no such splendor or such beauty, and we are at a loss how to describe it. We know only that God dwells here among men, and their service is fairer than the ceremonies of other nations. For we cannot forget that beauty.*

of their visit to the Hagia Sophia surely influenced the prince when he decided that his Russian state would embrace Byzantine Christianity.

After the city fell to the Ottoman Turks in 1453, the Hagia Sophia was converted into a mosque and its wonderful mosaics were plastered over. However, Plate 7 tells us that the interior of the Hagia Sophia was still a spectacular space 400 years thereafter.

Plate 7. The Hagia Sophia as a mosque. From Aya Sophia, Constantinople, as recently restored by order of sultan Abdul Medjid, from original drawings by Gaspard Fossati, lithograph by Louis Haghe, P.&D. Colnaghi&Co., London, 1852. The Marquard Library of Art and Archaeology, Princeton University Library, Albert M. Friend, 15 Bequest

Let's have a look at the basic structural aspects of the Hagia Sophia. Figure 3.2 shows a cross section of the church through the dome, half domes, and recesses. The shell of the dome is made of brick and mortar and is about $2\frac{1}{2}$ feet thick. Its inner and outer surfaces are sections of spheres that have the same center. Their circular cross sections and the common center are highlighted respectively in black (for the outer circle) and white (for the inner circle). Forty ribs radiate down from the top of the dome not unlike the ribs of an umbrella. Descending between the forty windows,

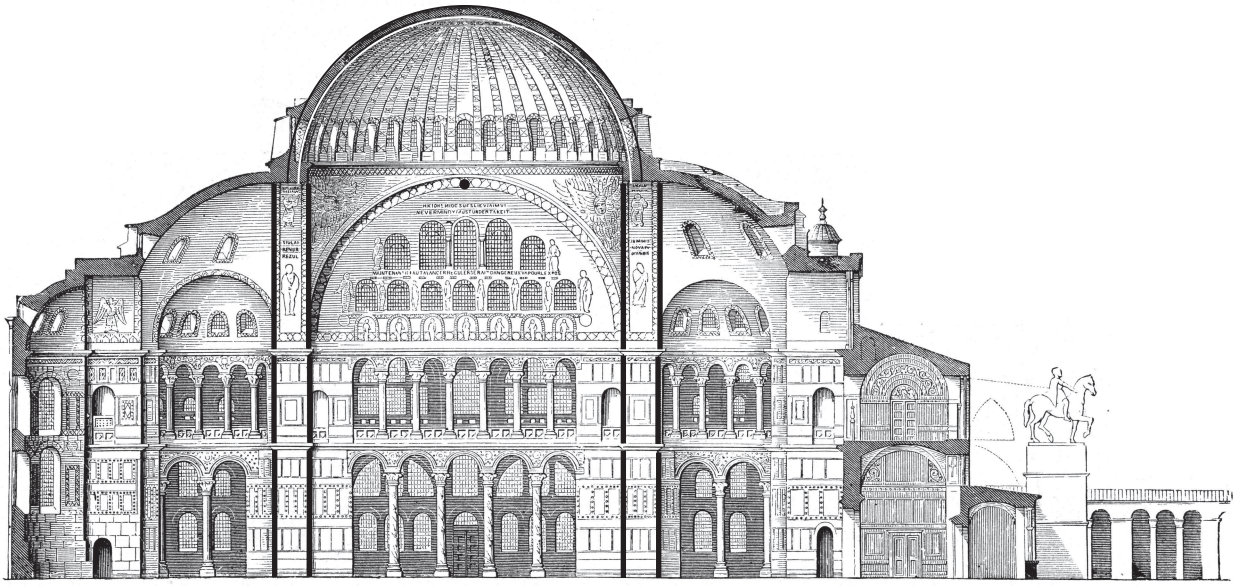

Figure 3.2. Section of the Hagia Sophia

they support the dome and anchor it to its circular base. The basic structural challenges facing the builders of the Hagia Sophia were the same as those faced by the Roman architects of the Pantheon four centuries earlier. The fact that brick and mortar have little tensile strength meant that the hoop stress on the shell generated by the downward push of the weight of the dome needed to be controlled by a strong supporting structure at the dome's base. As we saw in Section 2G, in the Roman Pantheon this structure is the massive, symmetric, closed cylinder from which its dome rose. It is apparent from the brief description already given that the design of the Hagia Sophia has a geometry that is more complex than that of a closed cylinder. Its dome rests on four large arches and the pendentives between them. Two of the arches open into half domes to form the long interior space of the church. The other two arches, as both Color Plate 6 and Figure 3.2 show, are closed off by walls that are perforated by rows of windows and arcades. So unlike that of the dome of the Pantheon, the support structure of the Hagia Sophia is asymmetric. This is problematic because it means that the dome of the Hagia Sophia was (and is) supported unevenly around its base.

Let's pause to consider the forces that the shell of the dome above the row of 40 windows generates. Figure 3.3 provides a detail of the cross section of the dome that is abstracted from Figure 3.2. The location of the windows, the ribs between them, as well as the supporting buttresses are shown. The two circular arcs are the cross sections of the inner and outer surfaces of the shell.

The information about the dimensions and building materials of the dome that follows is taken from recent studies. The two rays emanating from the common center of the two circles make an angle of about 20° with the horizontal. The inner and outer spherical surfaces of the shell have radii of

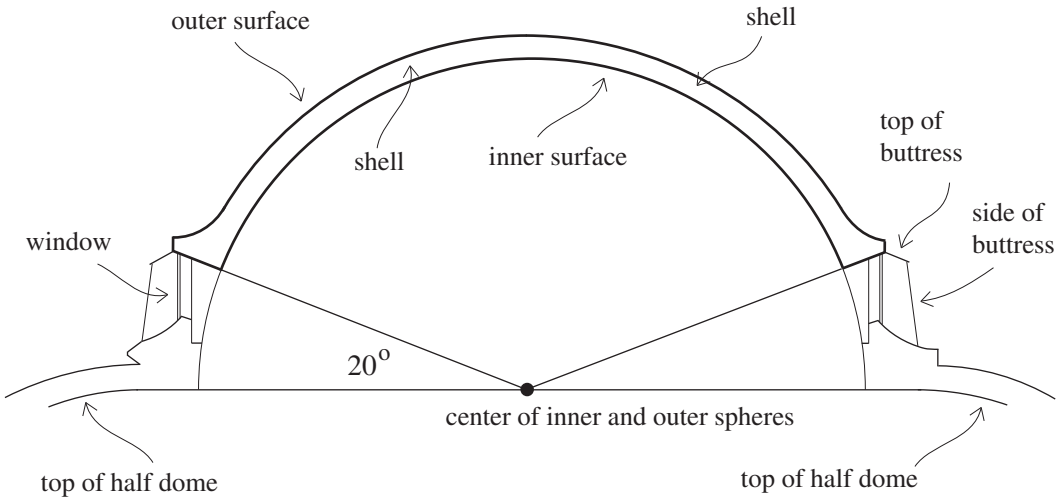

Figure 3.3

$r = 50$ feet and $R = 52.5$ feet respectively. The difference of 2.5 feet is the thickness of the shell. The average weight per cubic foot of the brick and mortar of the shell is about 110 pounds. Section 7B applies basic calculus to derive the estimate of 27,500 ft^3 for the volume of the shell of the dome above the circular gallery of windows. This implies that the weight of that part of the shell is approximately 27,500 $\text{ft}^3 \times 110$ lbs/$\text{ft}^3 \approx$ 3,000,000 pounds. Averaging this weight over the 40 supporting ribs, we get a load of about 75,000 pounds per rib. This means that if P is the slanting push by a rib, then the vertical component of P has a magnitude of about 75,000 pounds. It follows from Figure 3.4 that $\sin 70° = \frac{75{,}000}{P}$, so that therefore, $P \approx \frac{75{,}000}{\sin 70°} \approx$ 80,000 pounds. The horizontal

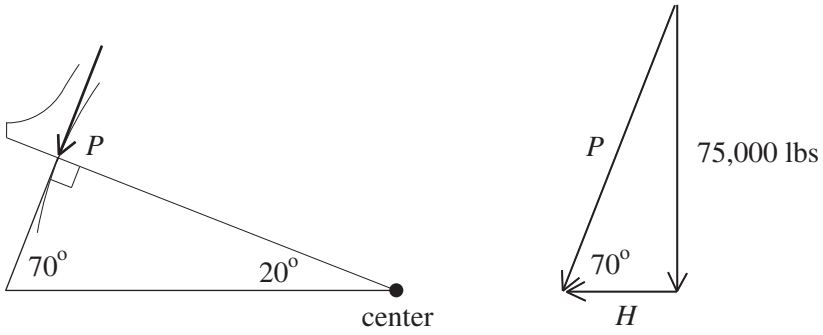

Figure 3.4

component H of the push P satisfies $\tan 70° = \frac{75{,}000}{H}$. Therefore $H \approx \frac{75{,}000}{\tan 70°} \approx$ 27,000 pounds. This is an estimate of the force with which a typical rib pushes outward against the base of the dome.

The architects of the Hagia Sophia were aware of the challenge that the outward thrust of the dome would present (although not in numerical terms) and they took measures to contain it. A rectangular roof structure that features four corners of heavy masonry above the pendentives braces

the dome at its base. It can be seen in Figure 3.5 below the circular array of windows. This structure and the four main arches are carried by the four stone piers already mentioned. They rise from foundations of solid rock. These piers are highlighted in black in Figure 3.2. The outward thrust of the dome is contained in two ways. In the long open direction, it is channelled downward and absorbed by the supporting half domes and the sloping structure beyond them (as shown in Figures 3.2 and 3.5). This is similar in principle to the way a Roman arch transfers its loads downward. In the perpendicular direction the outward thrust is contained by two great external arches under each side of the rectangular roof. One of these arches can be seen in Figure 3.5.

The Hagia Sophia has had a difficult history. The stresses that the building is under makes it particularly vulnerable to the earthquakes that are common in both Greece and Turkey. An

Figure 3.5. The Hagia Sophia as it looked in 1897. The four spires are minarets added after the conversion of the church to a mosque

earthquake led to a partial collapse of the dome only 20 years after its completion. By 563 the dome had been completely rebuilt about ten feet higher than before. This is the dome discussed above. It is still in place today. The 40 buttresses that brace the 40 ribs between the windows of the dome

were added at that time. They are visible as a dark rim in Figure 3.5. Additional earthquakes in the 10th and 14th centuries did major damage to the dome and extensive repairs were required each time. These repairs also responded to basic structural problems that had arisen over time. They included the correction of deformations of the main piers. The forces that the dome and the two great exterior arches generate in the direction of the two arches are absorbed by the half domes and the structures behind them. But the two great arches were deflected outward by the thrust of the dome and huge buttresses were added on the sides of the arches to stabilize them. Two of these buttresses can be seen in Figure 3.5. The structural elements that were added to contain the thrust of the dome do not intrude on the interior space of the church, but they do take a toll on its outward appearance. The mound-like exterior of the Hagia Sophia lacks the rising elegance of its interior.

As a consequence of the action of the stresses, the damage to the structure, and the extensive repairs that addressed them, the base of the dome is no longer a circle, but an oval. Figure 3.6 shows the horizontal cross section of the dome at its base. Notice that the diameter in the open direction of the two half domes is about $3\frac{1}{2}$ feet shorter than the diameter between the two great

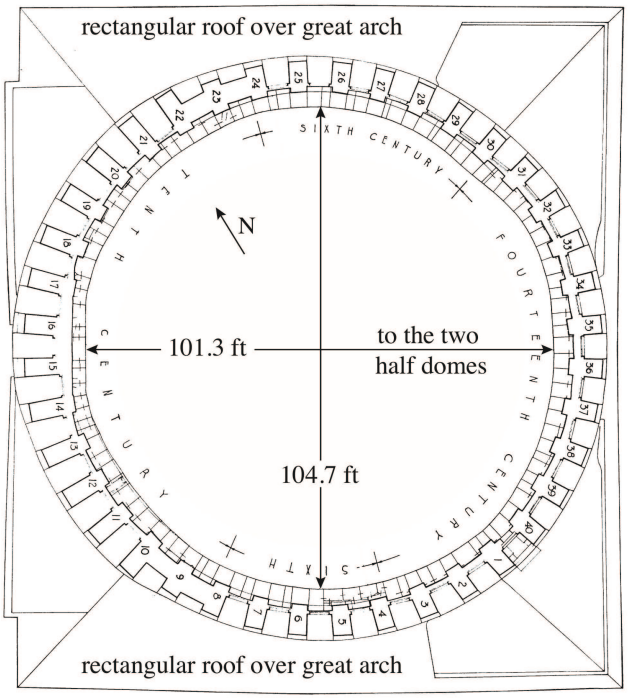

Figure 3.6. Section of the Dome of the Hagia Sophia at its base

exterior arches. This is consistent with both the stabilizing role played by the two half domes and the outward deflection that the great arches experienced. Figure 3.6 also provides the positions of the ribs – numbered from 1 to 40 – between the windows at the base of the dome and indicates when the various sections of the dome were repaired.

By the middle of the 19th century, the great building was once again in need of large scale repairs. The sultan of the time called upon the Fossati brothers, a pair of Swiss architects, to carry them out. To better contain the outward forces of the dome of the Hagia Sophia, the Fossatis placed an

iron ring around its base. This was a strategy that had already been used to brace the dome of St. Peter's in Rome. Color Plate 6 is a lithographic plate from a set of 25 plates fashioned by one of the brothers to record the results of the reconstruction. Today, 1500 years after it was built, the Hagia Sophia – converted to a museum in 1935 – is still a grand structure.

It has been said that the architects of the Hagia Sophia made use of mathematics in its design and its execution. While geometry clearly played a role, simple geometric considerations cannot give much information about the stability of a massive building. This was realized much later by Galileo who observed that geometry alone can never insure structural success. There is no evidence to suggest that applied mathematics was advanced enough at the time the Hagia Sophia was built to provide even the most elementary analysis of the loads that the structure would have to bear. There is little doubt that the architects relied on experience gained from earlier projects rather than on theoretical analyses.

Problems

Figure 3.39 depicts the vertical cross section of a shell of a dome much like that of the Hagia Sophia above its gallery of 40 windows. The inner and outer boundaries of the shell lie on concentric spheres. In Figure 3.39a, C is the common center of the two spheres, r is the radius of the inner sphere, and θ is the angle that determines the extent of the shell. The horizontal circular base of the dome along with its center are shown Figure 3.39b. The circular base has radius b and its distance from the top of the inside of the shell is a. The shell is assumed to have a rib structure

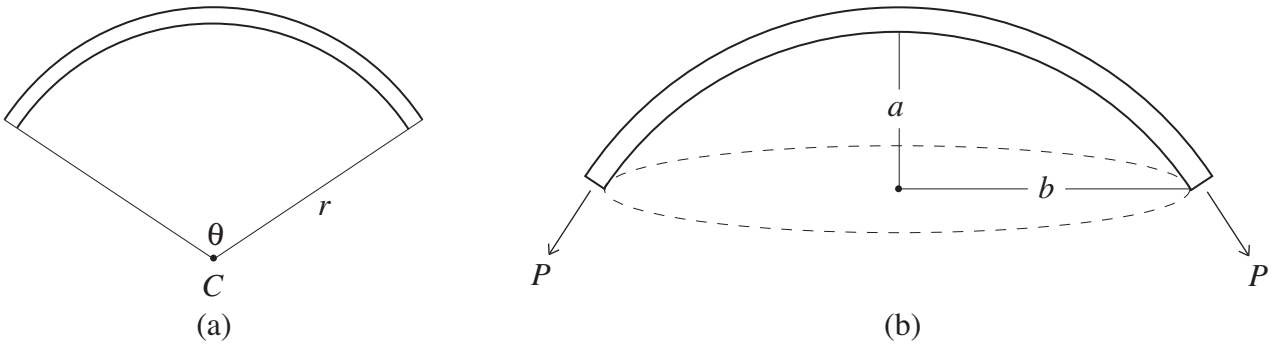

Figure 3.39

like that of the Hagia Sophia and the vectors labeled by P denote the push of two opposing pairs of ribs against the base of the shell.

Problem 1. Recall that for the dome of the Hagia Sophia $r = 50$ feet and $\theta = 140°$. Conclude that $b = 47$ feet and $a = 33$ feet.

An earthquake caused the partial collapse of the original dome of the Hagia Sophia soon after its construction had been completed and the dome was rebuilt. Not much seems to be known about the original dome other than the fact that it was lower and flatter than the rebuilt dome. However, under the assumption that the basic structure of the original dome was the same as that

of the rebuilt dome, it is possible to draw a number of speculative conclusions about it. These are developed in Problems 2 and 3 below. The assumptions made about the original dome are that Figure 3.39 depicts its essential structure, that the size of its circular base was the same as that of the rebuilt dome, and that it had a rib structure with 40 ribs. In view of the results of Problem 1, we will take the radius of the circular base of the original dome to be $b = 47$ feet and assume that the distance from this circular base to the top of the inside of the shell was $a = 23$ feet, 10 feet less than that of the rebuilt dome. The fact that the original dome was flatter means that the inner radius r of Figure 3.39 must have been larger.

Problem 2. Given the assumptions that have been made, show that for the original dome of the Hagia Sophia $r = 60$ feet and $\theta = 104°$ (both approximately). [Hint: Use the Pythagorean Theorem to find r. Then notice that $\sin\frac{\theta}{2} = \frac{b}{r}$.]

The fact that the shell of the dome of the Hagia Sophia is $2\frac{1}{2}$ feet thick made it possible to derive the estimate of 27,600 cubic feet for its volume. This volume computation is carried out in the section "Volumes of Spherical Domes" of Chapter 7. Assume that the original shell also had a thickness of $2\frac{1}{2}$ feet. The results of Problem 4 in combination with a similar volume computation provide the estimate of 23,300 cubic feet for the volume of the original shell.

Problem 3. Assume that the masonry of the original shell weighed the same 110 pounds per cubic foot as that of the rebuilt shell. Conclude that the original shell weighed approximately 2,560,000 pounds. Refer to the section "The Hagia Sophia" and derive the estimates $P \approx 81{,}000$ pounds for the push of one rib against the base of the original dome and $H \approx 50{,}000$ pounds for the horizontal component of P. Compare these estimates with those for the rebuilt dome and discuss the differences.

It is important to note that the basic underlying assumption of the study above—as well as the one that preceded it in the section "The Hagia Sophia"—is that the opposing pair of ribs shown in

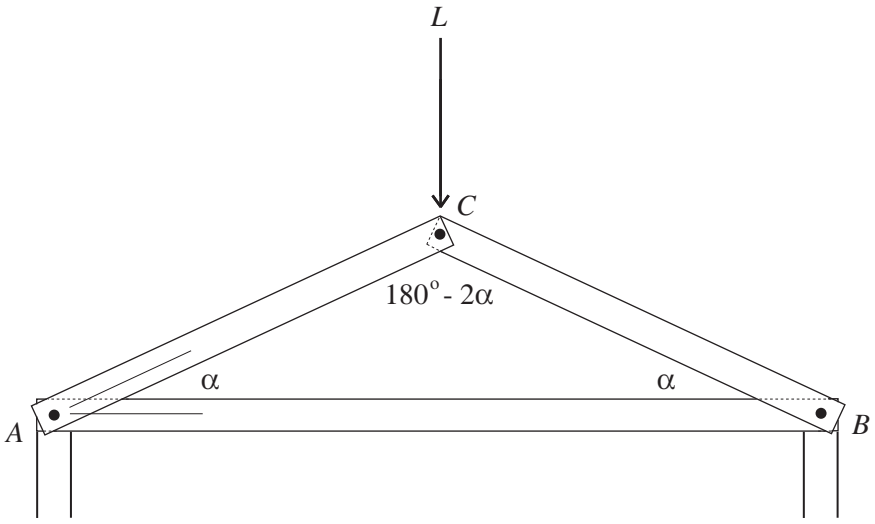

Figure 3.14

Figure 3.39b and the loads that the ribs carry are modeled by the simple truss depicted in Figure 3.14 and analyzed in the section "Romanesque Architecture."